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ABSTRACT: *This article reports on a study aiming to investigate an approach to the teaching and learning of proof based on the use of the transformation geometry tools of the software Cabri-Géomètre. We describe a teaching experiment involving students of the 7th and 8th grade of the Brazilian school system. The experiment involved the design and analysis of learning situations intended to involve students in inductive and deductive reasoning as they engage in analyses of both the (intra) properties within figures and the (inter) relationships between them. By considering students' interactions with these situations, we explore the role of the transformation tools in different aspects of the proving process, from the appropriation of notions of geometrical dependency to the construction of formally presented proofs.*

INTRODUCTION

In relation to the teaching and learning of geometry in Brazil in recent history, Pires *et al*. (2000) describes three distinct phases, leading up to the development of a set of parameters for a National Mathematics Curriculum of Brazil in 1998. In the first phase, characterising the mathematics curriculum before the impact of the Modern Mathematics movement, geometry teaching involved a traditional approach to Euclidean Geometry, in which proving activities were restricted to the memorisation of formal proofs. The introduction of curriculum materials based on Modern Mathematics had the effect of reducing the emphasis given to geometry: geometry was to be treated in the framework of group theory, an approach that was completely unfamiliar to the majority of mathematics teachers. During the 1980s, mathematics educators started to express grave concerns about the lack of emphasis on geometrical content in school mathematics and a new phase in which experimentation was prioritised began.

In 1998, when the parameters for the National Mathematics Curriculum of Brazil (PCN) were published (MEC, 1998)[1], geometry returned to represent a substantial content area, with Space and Form one of the four blocks of study included. In terms of the teaching and learning of proof, two aspects of the geometry curriculum are particularly relevant. The guidelines propose that, in addition to geometrical activities

[1] These parameters are intended to serve as guidelines and are not obligatory. They have had a considerable impact on the delivered curriculum, not least because the recommended textbooks have modified their schemes of study on the basis of the parameters.

involving exploration and experimentation, in the 7th or 8th grade[2] students should be expected have their first contact with proof and proving. What is intended is not a return to the traditional approach, students are expected, not to memorise and reproduce the formal proofs of others, but to engage in their own attempts to construct mathematical justification in ways coupled with, but eventually distinguished from, the use of empirical methods to verify (MEC, 1998; p.89).

One approach increasing explored in order that an appropriate balance between inductive and deductive reasoning may be achieved is the integration of dynamic geometry systems (DGS) into learning situations associated with proof (see, for example, Azarello *et al*., 1998; Gravina, 2000; Healy & Hoyles, 2001; Mariotti, 2001; Marrades & Gutiérrez, 2000). There seems to be general agreement that the use of DGS encourages users to become more cognisant of the geometrical properties and relations of the visual artefacts that they produce, but that the kinds of interactions with these artefacts that help learners justify why and when these relationships exist cannot be expected to emerge spontaneously.

All these studies have in common a focus on "classic" Euclidean geometry constructions and congruency of triangles as the principal tools for proof and, although geometrical transformations are available as construction tools in DGS, their potential role in the teaching and learning of proof seems not yet to have received much attention. This brings us to a second interesting aspect of the geometry curriculum in Brazil. It is suggested that students are encouraged to investigate congruency of figures in the plane through a study of the isometry transformations in order that geometry might be experienced in a dynamic rather than static manner.

THE STUDY

These considerations motivated a study into the constructions and justifications produced by students in learning situations involving the use of the transformation tools of the DGS *Cabri-Géomètre*. Our aim was to devise situations in which students would come into contact with ideas fundamental to proof. We wanted them to see how from a small collection of given properties – in our case, the properties incorporated in the isometry tools of the software – other geometrical properties necessarily emerge and we wanted to them to experience how by justifying the second set of properties the theoretical system with which they are working can be extended. In terms of task design, we were hence presented with a considerable

[2] Learners who progress through the Brazilian education system without repeating any years of study are aged between 12-14 years in the 7th and 8th grades.

challenge: to harness the potential of the Cabri-Géomètre microworld to come up with activities which (a) encourage students to focus on relationships between geometrical objects and (b) offer support for students to development arguments to explain why these relationships hold.

In this article, we describe our attempts to such design activities. We should point out from the start that we view the process of design as a fundamental part of the research process, necessitating a series of iterative cycles during which tools, tasks and teaching interventions are developed as students' activities with them are observed and analysed. In the following sections, we describe some of the critical decisions in this process and how they impinged on students' strategies and productions.

The learning situations we are developing are intended for students of the 7^{th} grade and 8^{th} grade and have been organised into three sets: *introduction* of the transformations reflection, translation and rotation; *identification* of properties associated with the use of the transformation tools; *construction* of given quadrilaterals using these tools, accompanied by *proofs* that the constructions guarantee the required properties. We have presently completed two cycles of the design phase. In the first cycle, data were collected as students worked in pairs with one of us, while in the second data were collected as two group of six students (a 7^{th} grade group and a 8^{th} grade group) again with one of the researcher in the role of teacher, working with three computers negotiated the activities in these sets. Students' discussions were captured in audio recordings and their computer constructions and written work were collected for analysis.

In addition to the literature related to students' proving activities, another tool for analysis was drawn from the work of Piaget and Garcia (1983), who suggest that major mathematical ideas pass through an ordered sequence of epistemological levels (which also characterise the historical development of mathematical knowledge). They proposed that ideas develop successively through three levels – intra, inter and trans, whereby attention moves from internal relationships defining objects, to relationships between them, then to structures into which internal and external relationships can be organised. As we are interested in transformations in their own right not in the set of isometries as a structured group, we concentrated analysis on movements between the intrafigural and interfigural levels. These two levels were used both to classify task demands and as a way of interpreting students' interactions with them.

From the identification of properties associated with the transformations…

The first set of activities introduced the `reflection`, `translation` and `rotation` tools of Cabri-Géomètre. It comprised three activities aiming to encourage students to focus on the properties of the geometrical designs produced using these tools. In each activity, students were given a starting figure and the elements necessary to apply a transformation tool (axes in the case of reflection, vectors for translation and a point and angle to apply a rotation). The task was to complete a design. Figure 1 presents examples of the designs produced.

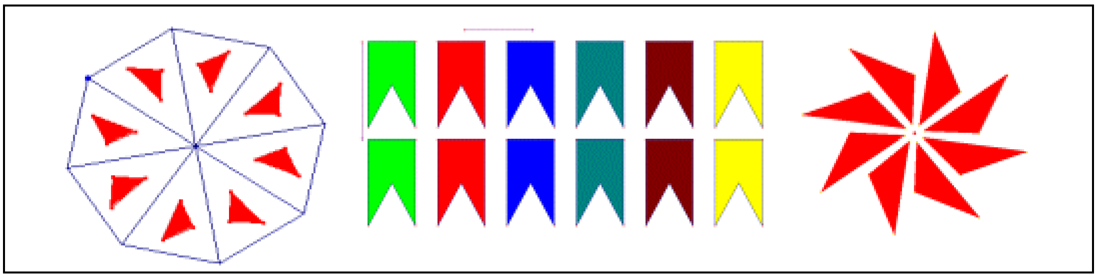
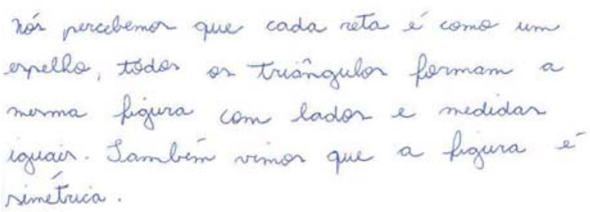


Figure 1: Designs produced during the introduction activities

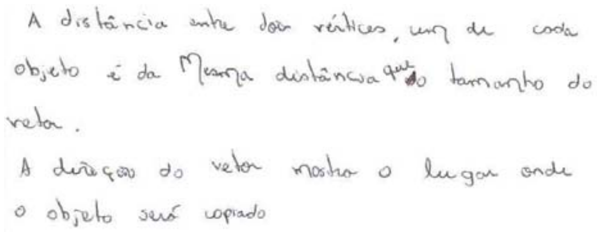
These activities privilege, in the main, intrafigural analyses, with students tending to focus on the internal properties of the final configuration displayed on screen rather than external relationships by which different figures could be related. The most common observation made by students related to the congruency of the transformed objects (intrafigural according to Piaget and Garcia, as only properties internal to figures are considered). Interfigural analyses were observed, however, in connection with the use of the `translation` tool. Of the six pairs observed in during the second cycle, for example, two pairs identified that the difference between the vertices of the flags was determined by the length of the vectors, while four mentioned the effect of the vector's direction. Figure 2 presents examples of student descriptions with respect to the reflection and translation activities.

The second set of activities was designed to emphasise geometrical relationships more explicitly. It composed of four activities. The first activity explored again the conservation of within-figure distances and angles associated with the transformations reflection, translation and rotation in turn. The second and third activities involved identifying conditions under which image-figures parallel or perpendicular to corresponding pre-image figures could be produced, again with each transformation considered in turn. The fourth activity focussed students' attention on the conservation of the distances between objects associated with each transformation.


nós percebemos que cada reta é como um espelho, todos os triângulos formam a mesma figura com lados e medidas iguais. Também vimos que a figura é simétrica.

"We noticed that each line is like a mirror, all the triangles form the same figure with equal sides and measures. Also we saw that the figure is symmetrical".

A: Intrafigural-orientated observations of Letícia and Priscila (8th grade)


A distância entre dois vértices, um de cada objeto é da mesma distância que do tamanho do vetor.
A direção do vetor mostra o lugar onde o objeto será copiado

"The distance between the vertices, one from each object is the same distances as the size of the vector. The direction of the vector shows the place where the object will be copied."

b: Interfigural-orientated observations of Guilherme and Paula (8th grade)

Figure 2: Students' observations during the first activity set

It is important to reiterate at this point that in our study it is the properties embedded in the isometry tools of Cabri-Géomètre that serve as the starting point for proof, that is, they define our theoretical reference system and can be treated by students as facts that do not require justification. From a mathematical point of view, the invariants associated with the isometries are conservation of distance between points, conservation of angles and conservation of alignment of points. The properties observed by students as they worked on the activities are consequences of these three postulates (in the Euclidean context of Cabri-Géomètre). We are hence treating the transformation tools as "geometrical primitives" (Laborde, 1993) in a deductive system.

Analysis of students' discussions during the second activity set indicates that all of the students were clear that figures congruent to the original result from all three transformations. As the extract from a discussion between two students and the researcher reproduced below illustrates, both intrafigural and interfigural interpretations permeated their interactions. Henrique e Lylli use the word "symmetry" to describe the three transformations. It may be that symmetrical for them is synonymous with congruent. In any case, they seem to have appropriated a feature common to the three transformation tools (they produce "copies" of figures) as well as beginning to discern features by which they can be distinguished – thus combining intra and interfigural concerns.

Hen: *It's these things here* [the isometry tools], *when you do them, you just take the original figures and modify it in space, like the space that it is, but it stays the same figure...*

Lylli: *...They are all types of symmetry. This* [reflection] *is a type of mirror symmetry, this here* [translation] *is symmetry moving in space, this is the difference in space* [pointing to the vector]...*here is where it shows up or down or the direction.*

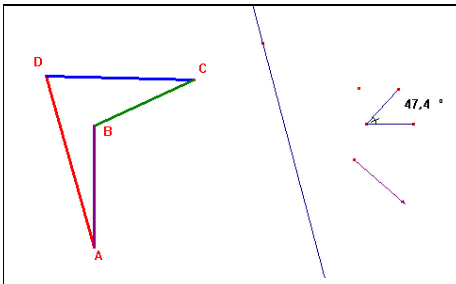
Res: *So your saying they are all, translation, reflection and rotation, are all types of symmetry?*

Lylli: *Yes. Because* [rotation] *is like you have a copy of this triangle and you turn it.*

Hen: [laughing] *One is symmetric symmetry, another is rotationary symmetry and one is translationary symmetry.*

In addition to identifying that distances and angles are invariant under the three transformations, we also wanted students to think about their relationships with parallelism and perpendicularism. That is, that segments transformed by translation are always parallel to pre-image segments, while, in the case of reflection and rotation, parallel (or perpendicular) segments result from specific configurations of the elements defining the transformation. To check (empirically) if the parallel or perpendicular conditions are satisfied, students were introduced to the tools `parallel?` and `perpendicular?` To a certain extent, we were aiming to define these two properties in terms of the transformations.

Students had very little difficulty in determining that when the `translation` tool is employed, images of segments are invariably parallel to the corresponding pre-image segments (Figure 3 shows how this task was presented to the students). Over half the student pairs used this result to deduce that segment and image under translation are never perpendicular, the rest confirmed this empirically.



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Using the translation tool, a segment *AB* of the original polygon is parallel to its image *A'B'*

...... always

...... never

......sometimes, when ....

Figure 3: Determining conditions for obtaining parallel segments (translation)

Similarly, it did not seem to be hard for students to locate the angles in rotations necessary to produce parallel or perpendicular images segments – although some pairs did not indicate all the possible values of the angle of rotation. In contrast, it turned out to be more difficult to determine the conditions under which parallel and perpendicular image segments are obtained by reflection. If students restricted their activities to manipulating a general axis, they were able to successfully identify the necessary orientation of the axis in the case of a vertical places segment, but did not describe the parallel relationship between axis and segment that holds for any orientation of *AB*. In contrast, in the second cycle, an intervention suggesting the axis was constructed, enabled students to identify the necessary conditions for producing parallel and perpendicular segments, with the cost that they became somewhat reliant on the researcher. One challenge for the next cycle is to rethink the reflection activities in a way that allows students to see general relationships without losing control of the solution process.

… To construction and proof

The third set of activities involves the development of construction methods for familiar quadrilaterals – squares, rhombi, parallelograms and rectangles – using the transformation tools. We were hoping that they would distinguish between the properties directly constructed and those that emerged as a result of the construction process, then make use of the properties identified during the first two activities-sets (by this point displayed on a table) to build arguments justifying the relationships between the property sets.

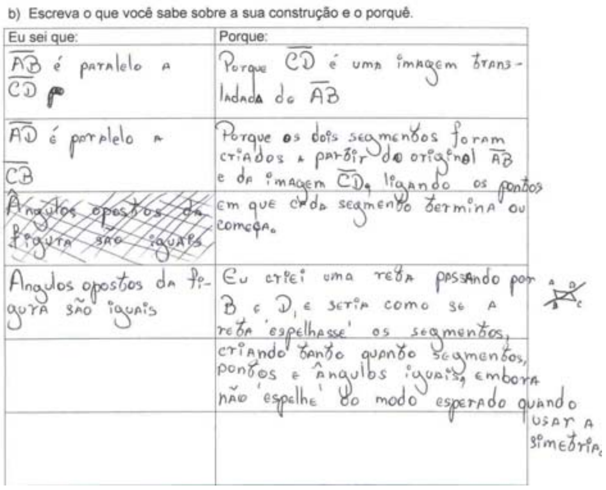
During the first cycle, it became clear that students needed some help in understanding what was involved in organising a proof. Once again, we intervened in the second cycle, providing students' with a sequence of nine steps (statements) and justifications representing a (fictional) student's proof that his construction had the properties of a square. The steps were organised in the correct order, but the justifications – largely based on the properties the students had identified – were not. The task was to match the justifications to the appropriate statement. Only one of the six pairs who tackled the activity matched all nine justifications correctly. Three pairs managed seven justifications, one pair five, and one pair (from the 8^{th} grade) correctly matched just two justifications to the appropriate statements. These results suggest that the proof was rather too long to serve as the first example.

The given square construction could also be used as a basis to build a robust rhombus and all the students made use of this construction procedure. The strategy of providing an example of the work of a fictional student hence had a marked effect on students' interactions, essentially solving the rhombus construction. However, this allowed students a further opportunity to make sense of the given justifications and organise them into a proof. Of the six pairs who participated in the second cycle, three pairs (two from the 7^{th} grade and one from the 8^{th} grade), constructed proofs we considered correct, two groups correctly justified all the properties used in the construction procedure but included properties not constructed as justifications in the final steps, and one pair produced a list a correct statements matched with correct justifications but presented in no particular order. These results suggest that the students were beginning to engage in the process of proving, although they had not yet appropriated all of the "rules" of the proving discourse.

When it came to the rectangle and parallelogram constructions, students could no longer rely on the support of a given construction and had to come up with proofs of their own. We describe the results with respect to the parallelogram. Most pairs started by constructing two segments joined at one vertex, after experimenting with reflection, some went directly to translation, while other experimented first with the `rotation` tool. Although it is possible to construct a parallelogram using rotation, this involves co-ordinating two independent elements, centre and angle of rotation, perhaps it was because of this complexity that student pairs left the tool aside in favour of translation. Without exception, all students made use of a vector external to the figure under construction to translate a segment, rather than choosing to define one of its sides as the vector. By dragging the vector to be parallel and of equal length to the second side, those who began with two segments were able to “close” their parallelogram – a soft construction. The only pair who managed a robust construction had begun with only one segment on screen, having translated this segments they simply joined the vertices of the pre-image segments to their images.

Considering the student proofs associated with this task, none of the student-pairs came up with a complete proof, but in all of the arguments presented, they included at least some correct statements and justification, which the majority attempted to organise into a logical chain. For example, Figure 4 presents the attempt of Henrique and Lylli, the 7th grade students who produced the only robust parallelogram. They started with the properties they had constructed (*AB* as parallel to *CD* by translation), and went on to try to explain the properties that resulted from their construction process. However, although they were correct in stating that the segments *AD* and *CB* are parallel, they did not justify why. They could have done so by explaining the relationship between these segments and the vector – an interfigural interpretation – in fact, not one student explicitly referred to this property in their attempts to justify, suggesting at the moment of proof, students are still focussed on the relationships within figures rather than with objects external to them.

The third statement included in the proof of Henrique and Lylli is particularly interesting. They – without any outside intervention – added a line to their parallelogram, in order to obtain two congruent triangles, reasoning that if these triangles were congruent (equal), then opposite angles in the figure must also be equal. They argue that the line they have added could be seen a mirror, mirroring segments and angles, although “*not in the way expected by a mirror*”. They wanted, as another member of the 7th grade group put it, a “*kind of inverted reflection*”. Rotation could have served this purpose, but perhaps we should have introduced students to the tool `symmetry`, which produces a rotation of 180º about a point (reflection in a point), before expecting them to work with the more general rotation tool.


b) Escreva o que você sabe sobre a sua construção e o porquê.

Eu sei que:	Porque:
$\overline{AB}$ é paralelo a $\overline{CD}$	Porque $\overline{CD}$ é uma imagem translada do $\overline{AB}$
$\overline{AD}$ é paralelo a $\overline{CB}$	Porque os dois segmentos foram criados a partir do original $\overline{AB}$ e da imagem $\overline{CD}$, ligando os pontos em que cada segmento termina ou começa.
~~Angulos opostos da figura são iguais~~	
Angulos opostos da figura são iguais	Eu criei uma reta passando por B e D, e seria como se a reta 'espelhasse' os segmentos, criando tanto quanto segmentos, pontos e ângulos iguais, embora não 'espelhe' do modo esperado quando usar a simetria.

I know that:	Because:
$\overline{AB}$ is parallel to $\overline{CD}$	Because $\overline{CD}$ is a translated image of $\overline{AB}$
$\overline{AD}$ is parallel to $\overline{CB}$	Because the two segmentos were created from the original $\overline{AB}$ and the image $\overline{CD}$, joining the points at
	which each segments ends or begins
Opposite angles of the figure are equal	I created a line passing through B and D and it would as if the line 'mirrored' the segments, creating equal
	segments, points and angles, although its doesn't 'mirror' in the mode expected when symmetry (reflection)[3] is used

Figure 4: Proving properties of a parallelogram

REASONING WITH TRANSFORMATIONS OR TRANSFORMATIONAL REASONING?

Reflecting on the students' interactions during both cycles of the study, we suggest that an approach to proving based on geometrical transformation in a dynamic geometry context holds some promise in supporting learners to begin to engaging in the complex process of proof. Our results suggest that the transformation tools can provide an accessible introduction to the notion of geometrical dependency, allowing students to experience visually and physically how the behaviour of some geometrical objects can be constructed to depend on their relationship to others. Not all the students however, succeeded in building constructions that were entirely robust using the transformations, although they did manage to use the transformation tools to set up appropriate properties.

Success on the activities depended on the development of strategies involving movement between intra and interfigural analysis. In the identification activities, students seemed able to do this. But, although in their computer constructions students engaged in interfigural interpretations, these were not always incorporated in their attempts to formulate valid justifications. Nonetheless, as the proof attempt in Figure 4 illustrates, the transformation tools permitted, simultaneously, static and dynamic interpretations of the quadrilaterals produced. This double perspective

[3] In the version of Cabri-Géomètre with which the students were interacting, "reflection" has been translated as "simetria axial", we saw above that students used the word "simetria" (symmetry) more generally to include situations involving the other isometries, but in this case it appears that they were referring to the transformation associated with the `reflection` tool.

resulted in arguments based on general properties and their relationship rather than specific observations, although the validity of the students' arguments varied according to the task and construction methods employed.

Our analysis also led us to reflect about a third type of reasoning, not inherently inductive or deductive, to which mathematics education researchers are currently attributing an important role – transformational reasoning (Simon, 1996; Harel & Sowder, 1998; Arzarello & al. 1998, Gravina, 2000). According to Simon (1996), transformational reasoning is:

> "…mental or physical enactment of an operation or a set of operations on an object or a set of objects that allows one to envisage the transformations that these objects undergo and the set of results of these operations. Central to transformational reasoning is the ability to consider, not a static state, but a dynamic process by which a new state or continuum of states are generated." (p.201)

We suggest that perhaps Simon is not referring to a (new) form of reasoning with a particular role in relationship to proof, but that his definition is characteristic of mathematical analyses involving both interfigural and intrafigural perspectives we were trying to promote in our experiment. This is a possibility we intend to explore as our research continues.

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